Stat 821 Homework 5 Solution

Question 1 (Problem 1.3)

(a) $P(X=i) \geq 0, \quad \forall\, i=-1,0,1,2,3$

$$\sum_{i=-1}^{3} P(X=i) = 2pq + q^3 + pq^2 + p^2q + p^3 = (p+q)^2 = 1$$

Let

$$\delta_0 = \begin{cases} 1/2 & k=-1 \\ 1 & k=2,3 \\ 0 & o.w. \end{cases}$$

Then

$$E(\delta_0) = \frac{1}{2}2pq + p^2q + p^3 = p$$

Unbiased!

Let

$$EU(x) = 0 \;\Rightarrow\; 2U(-1)pq + q^3U(0) + q^2pU(1) + qp^2U(2) + p^3U(3) = 0$$

$$\begin{aligned} \Rightarrow \quad & 2U(-1)p - 2U(-1)p^2 + U(0) - 3pU(0) + 3U(0)p^2 - p^3U(0) \\ & pU(1) - 2p^2U(1) + p^3U(1) + p^2U(2) - p^3U(2) + p^3U(3) = 0 \end{aligned}$$

$$\Rightarrow \quad U(0)=0 \quad U(-1)=a \quad U(1)=U(2)=-2a \quad U(3)=0$$

$$Var(\delta_0+U) = (\frac{1}{2}+a-p)^2 2pq + p^2q^3 + (2a+p)^2pq^2 + (1-2a-p)^2p^2q + (1-p)^2p^3$$

$$\frac{\partial}{\partial a^2} Var(\delta_0+U) = 0 \;\Rightarrow\; a_0 = \frac{2p-1}{6}$$

At $p_0, U_0 = (a_0, 0, -2a_0, -2a_0, 0)$

$$\frac{\partial^2}{\partial a^2} Var(\delta_0+U_0) = 12p_0q_0 > 0$$

Thus the LMVUE at p_0 is

$$\begin{cases} \frac{P_0+1}{3} & k=-1 \\ 0 & k=0 \\ -\frac{2p-1}{3} & k=1 \\ \frac{4-2p}{3} & k=2 \\ 1 & k=3 \end{cases}$$

There's no UMVUE since U is a function of p.

(b) Let

$$\delta_1 = \begin{cases} 1/2 & k=-1 \\ 0 & o.w. \end{cases}$$

Then

$$E\delta_1 = pq$$

$$Var(\delta_1+U) = (\frac{1}{2}+a-pq)^2 2pq+(pq)^2q^3+(2q+pq)^2q^2p+(2a+pq)^2qp^2+(pq)^2p^3$$

$$\frac{\partial}{\partial a}Var(\delta_1+U) = 2pq(1+2a-2pq)+4(2a+pq)pq = 0 \;\Rightarrow\; a = -\frac{1}{6}$$

$$\frac{\partial^2}{\partial a^2}Var(\delta_1+U) = 12pq > 0$$

Thus the UMVUE and LMVUE at any p is

$$\begin{cases} 1/3 & x=-1 \\ 0 & x=0 \\ 1/3 & x=1 \\ 1/3 & x=2 \\ 0 & x=3 \end{cases}$$

Question 2 (Problem 1.13)

Since δ_1, δ_2 in Δ, then $E(\delta_1^2) < \infty$, $E(\delta_2^2) < \infty$.

$$E(a_1\delta_1 + a_2\delta_2)^2 = a_1^2 E\delta_1^2 + a_2 E\delta_2^2 + 2a_1a_2E(\delta_1\delta_2)$$

$$E(\delta_1\delta_2) \le \sqrt{E\delta_1^2 E\delta_2^2} < \infty$$

Thus

$$E(a_1\delta_1 + a_2\delta_2)^2 < \infty \qquad a_1\delta_1 + a_2\delta_2 \in \Delta$$

δ_1, δ_2 are UMVUE for $g_1(\theta)$ and $g_2(\theta)$.

$$Cov(\delta_1, U) = Cov(\delta_2, U) = 0$$

where

$$u \in \mathcal{U} = \{\text{unbiased estimator for } 0\} \cap \Delta$$

$$Cov(a_1\delta_1 + a_2\delta_2, U) = a_1 Cov(\delta_1, U) + a_2 Cov(\delta_2, U) = 0$$

Thus $a_1\delta_1 + a_2\delta_2$ is UMVUE for $a_1 g_1(\theta) + a_2 g_2(\theta)$.

Question 3 (Problem 1.22)

(a)

$$\begin{aligned} E(\delta^*) &= E[E(\delta^*|X)] \\ &= E[I_{(x=0)}E(Y_0) + I_{(x=1)}E(Y_1)] \\ &= EI_{(x=1)} \\ &= p \end{aligned}$$

Thus δ^* is unbiased.

(b)

$$E[L(p, X)] = 1 \cdot P\{|X - p| \ge 1/4\} = P\{X = 0 \text{ or } X = 1\} = 1$$

since $(0-p) \in (-3/4, -1/4)$, $(1-p) \in (1/4, 3/4)$.

$$\begin{aligned} E[L(p,\delta^*)] &= 1 \cdot P\{|\delta^* - p| \geq \frac{1}{4}\} \\ &= qP\{|Y_0 - p| \geq \frac{1}{4}\} + pP\{|Y_1 - p| \geq \frac{1}{4}\} \\ &= q\int_{-1/2}^{-1/4+p} 1\,du + p\int_{1/4+p}^{3/2} 1\,du \\ &= -2(p-\frac{1}{2})^2 + \frac{3}{4} \leq \frac{3}{4} \end{aligned}$$

Thus $R(\delta^*, p) < R(X, p)$.

Question 4 (Problem 2.12)

(a) Suppose $X_1, \ldots, X_m$ and $Y_1, \ldots, Y_n$ are independent.

$$X_i \sim N(\xi, \sigma^2) \qquad Y_i \sim N(\eta, \tau^2)$$

$$\eta = \xi \quad \frac{\sigma^2}{\tau^2} = \gamma \quad \sigma^2 = \tau^2\gamma$$

$$\begin{aligned} & f(X, Y|\eta, \tau^2) \\ = \quad & \frac{1}{(2\pi)^{\frac{m+n}{2}}\gamma^{m/2}\tau^{m+n}} \exp\left[-\frac{\sum x_i + \gamma\sum y_i^2}{2\gamma\tau^2} + \frac{(\sum x_i + \gamma\sum y_i)\eta}{\gamma\tau^2}\frac{(m+\gamma n)\eta^2}{2\gamma\tau^2}\right] \end{aligned}$$

This belongs to full rank exponential family and hence T' is complete sufficient.

(b)

$$E\delta_\gamma = \alpha E\bar{X} + (1-\alpha)E(\bar{Y}) = \alpha\xi + (1-\alpha)\eta = \xi$$

Thus δ_γ is unbiased for ξ and is a function of T'.

$$\delta_\gamma = \alpha\bar{X} + (1-\alpha)\bar{Y} = \frac{1}{n\gamma + m}(\sum x_i + \gamma\sum Y_i)$$

Thus δ_γ is UMVUE of ξ.

Question 5 (Problem 2.19)

If $b = 1$, we can write 2.22 as $\exp\{-\sum(x_i - a)\}I_{(a,\infty)(x_{(1)})}$

$$P(X_1 \geq u) = \begin{cases} e^{-(u-a)} & u > a \\ 1 & u \leq a \end{cases}$$

$X_{(1)}$ is sufficient for a and has pdf

$$f(x_{(1)}) = ne^{-n(x_{(1)}-a)}I_{[a,\infty)}(x_{(1)})$$

$$E[g(X_{(1)})] = 0 \ \Rightarrow \ \int_a^\infty g(x_{(1)})e^{-nx_{(1)}}dx_{(1)} = 0$$

$$\frac{d}{da}\int_a^\infty g(x_{(1)})e^{-nx_{(1)}}dx_{(1)} = -g(a)e^{-na} = 0 \ \Rightarrow \ g(a) = 0 \ a.e.$$

Thus $X_{(1)}$ is complete for a.

$$E[I(X_1 \geq u)] = I_{(u \leq a)} + e^{-(u-a)}I_{(u>a)}$$

Then the UMVUE is $E(I(X \geq u)|X_{(1)})$. Now we need to find the conditional distribution of $X_i|X_{(1)}$.

$$P(X_1 = X_{(1)}|X_{(1)}) = \frac{1}{n}$$

For $X \neq X_{(1)}$,

$$\begin{aligned} P(X_1 = x|X_{(1)}) &= \frac{P(X_1 = x, X_{(1)} = x_1)}{P(X_{(1)} = x_1)} \\ &= \frac{nf(x_{(1)})f(x)[1-F(x_{(1)})]^{n-2}}{ne^{-n(x_{(1)}-a)}} \\ &= e^{-(x-x_{(1)})} \end{aligned}$$

And

$$P(X_1 \neq x_{(1)}|X_{(1)} = x_{(1)}) = \frac{n-1}{n}$$

Thus

$$E(I(X_1 \geq u)|X_{(1)}) = P(x_1 \geq u|X_{(1)}) = \begin{cases} 1 & u \leq X_{(1)} \\ \frac{n-1}{n}e^{-(u-x_{(1)})} & u > X_{(1)} \end{cases}$$

is the UMVUE of $P(X_1 \geq u)$.

$$P(X = u) = E[I_{(X=u)}] = e^{-(u-a)} \quad \text{for } u \geq a$$

Then $I_{(X=u)}$ is unbiased estimator of $e^{-(u-a)}$. $E(I_{(X=u)}|X_{(1)})$ is the UMVUE.

$$E(I_{(X=u)}|X_{(1)}) = P(X = u|X_{(1)}) = \begin{cases} \frac{1}{n} & u = x_{(1)} \\ \frac{n-1}{n}e^{-(u-x_{(1)})} & u > x_{(1)} \\ 0 & u < x_{(1)} \end{cases}$$

Thus the UMVUE of $e^{-(u-a)}$ is

$$\frac{1}{n}I_{(u=X_{(1)})} + \frac{n-1}{n}e^{-(u-X_{(1)})}I_{(u>X_{(1)})}$$

Question 6 (Problem 2.25)

$$X_{(m)} = \max X_i \quad \text{is complete sufficient for } \theta$$

$$Y_{(n)} = \max Y_i \quad \text{is complete sufficient for } \theta'$$

$$E[X_{(m)}] = \frac{m}{m+1}\theta \quad \Rightarrow \quad \frac{m+1}{m}X_{(m)} \text{is UMVUE for } \theta$$

$$E[1/Y_{(n)}] = \frac{n}{(n-1)\theta'} \quad \Rightarrow \quad \frac{n-1}{nY_{(n)}} \text{is UMVUE for } 1/\theta'$$

Thus

$$E\left[\frac{(m+1)X_{(m)}}{m}\frac{(n-1)}{nY_{(n)}}\right] = E\left[\frac{(m+1)X_{(m)}}{m}\right]E\left[\frac{(n-1)}{nY_{(n)}}\right] = \frac{\theta}{\theta'}$$

$$\Rightarrow \quad \frac{(m+1)(n-1)X_{(m)}}{mnY_{(n)}} \quad \text{is unbaised}$$

Next, we prove it is UMVUE. Let

$$\delta_1 = \frac{(m+1)X_{(m)}}{m} \quad \delta_2 = \frac{(n-1)}{nY_{(n)}} \quad U \in \mathcal{U} = \{\text{unbiased estimator of zero}\}$$

$$
\begin{aligned}
Cov(\delta_1\delta_2, U) &= E(\delta_1\delta_2 U) - E(\delta_1\delta_2)E(U) \\
&= E(\delta_1\delta_2 U) \\
&= E[E(\delta_1\delta_2 U|\delta_1)] \\
&= E[\delta_1 E(\delta_2 U|\delta_1)] \\
&= E[\delta_1 E(\delta_2)E(U|\delta_1)] \qquad \text{(since } \delta_2 \text{ and } \delta_1 \text{ are independent)} \\
&= E[\delta_1 E(\delta_2)E(U)] \qquad \text{(since } U \text{ and } \delta_1 \text{ are independent)} \\
&= 0
\end{aligned}
$$

Therefore,

$$
\frac{(m+1)(n-1)X_{(m)}}{mnY_{(n)}} \quad \text{is UMVUE}
$$